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Approximation of frictional contact for plates using Nitsche's method

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Résumé — We describe new Nitsche's methods extensions applied to unilateral contact for plates. Our analysis is based on the interpretation of Nitsche's 3D method with kinematic assumptions of Kirchhoff-Love or Mindlin–Reissner theories. To simplify the presentation, we focus on the contact for an elastic plate and a rigid obstacle. We end the paper by presenting and comparing results of our numerical computations which corroborate the efficiency and reliability of our approach.
Mots clés — Nitsche methods, unilateral contact, plates theories.

1 Notations and statement of the problem

1.1 Introduction

The Nitsche method orginally proposed in [9] aims at treating the boundary or interface conditions in a weak sense, according to the Neumann boundary operator associated to the partial differential equation and in a consistent formulation. In [3, 8, 12] Nitsche's method has been extended to discretize contact and friction conditions. It differs in this aspect from standard penalization techniques which are generally non-consistent [1, 2, 4]. Moreover, no additional unknown (Lagrange multiplier) is needed and no discrete inf-sup condition must be fulfilled, contrarily to mixed methods. The purpose of this note is to present error analysis of conforming finite element methods for classical 2D or 3D plate bending models, as initiated in [11].

Let us consider a thick or a thin elastic plate *i.e.* a plane structure for which one dimension, called the thickness, is small or very small compared to the others. For this kind of structures, starting from *a priori* hypotheses on the expression of the displacement fields, a two-dimensional problem is usually derived from the three-dimensional elasticity formulation by means of integration along the thickness. Then, the unknown variables are set down on the middle plane of the plate.

1.2 Kinematic of the plate models

Let Ω be an open, bounded, connected subset of the plane $\mathbb{R}^2$, with Lipschitz boundary. It will define the middle plane of the plate it is soppose to be in the plane $x_3 = 0$, in reference configuration. Then, the plate in its stress free reference configuration coincides with domain :

$$\Omega^{\varepsilon} \;=\; \Omega \times \,]-\varepsilon\,,\,+\varepsilon[\,=\,\left\{\,(x_1,x_2,x_3) \in \mathbb{R}^3 \;/\; (x_1,x_2) \in \Omega \text{ and } x_3 \in \,]-\varepsilon\,;\,\varepsilon[\;\right\},$$

where $2\,\varepsilon > 0$ is called the thickness. For a plate made of a elastic, homogeneous and isotropic material, which mechanical constants are its Young's modulus E, its Poisson's ratio N. As usual, we have : $E \;>\; 0$, $0 \;\leq\; N \;\leq\; 0.5$.

Moreover, δ_{ij} is the Kronecker's symbol and the summation convention over repeated indices is adopted, Greek indices varying in $\{1,2\}$ and latin indices varying in $\{1,2,3\}$. In the following ∂_i stands for the partial derivative with respect to x_i and the second derivatives are $\partial^2_{ij} \;=\; \dfrac{\partial^2}{\partial x_i\,\partial x_j}$.

In general plate theory, it is assumed that a mid- plane surface plane can be used to represent the three-dimensional plate in two-dimensional form. Thus it is usual to consider the following (first order) linear approximation of the three-dimensional displacements for $x = (x_1, x_2, x_3) \in \Omega^\varepsilon$:

$$\left\{ \begin{array}{lcl} u_1(x_1,x_2,x_3) & = & \underline{u}_1(x_1,x_2) + x_3\, \theta_1(x_1,x_2), \\ u_2(x_1,x_2,x_3) & = & \underline{u}_2(x_1,x_2) + x_3\, \theta_2(x_1,x_2), \\ u_3(x_1,x_2,x_3) & = & \underline{u}_3(x_1,x_2). \end{array} \right. \tag{1}$$

In these expressions, $\underline{u}_1$ and $\underline{u}_2$ are the membrane displacements of the mid-plane points, $\underline{u}_3$ is the deflection, while θ_1, θ_2 are the section rotations.In the case of an homogeneous isotropic material, the variational plate model splits into two independent problems : the first, called the membrane problem, deals only with membrane displacements, while the second, called the bending problem, concerns deflection and rotations. In the theory of thick plates, or Mindlin-Reissner model, the normal to the mid-surface remains straight but not necessarily perpendicular to the mid-surface. In this paper, we shall also consider the Kirchhoff-Love model, which can be seen as a particular case of (1) obtained by introducing the so-called Kirchhoff-Love assumptions :

$$\left\{ \begin{array}{lcl} \theta_1 & = & -\partial_1\, u_3, \\ \theta_2 & = & -\partial_2\, u_3, \\ \theta_3 & = & 0. \end{array} \right. \tag{2}$$

Consequently, the deflection is the only unknown for the bending Kirchhoff-Love plate problem and this displacement is independent of membrane displacements (i.e in-plane displacements of the mid-plane surface).

1.3 3D-Contact problems : variational formulations

1.3.1 Loading and bilateral boundary conditions imposed

In the same way as with the general three-dimensional solid, the boundary $\partial\Omega^\varepsilon$ of the domain Ω^ε is assumed to be partitioned into three parts. In line with the particular shape of the domain, these parts are now the "lateral" one Γ_0^ε, the "upper" one Γ_+^ε and the "lower" one Γ_-^ε, i.e. :

$$\partial\Omega^\varepsilon = \Gamma_0^\varepsilon \cup \Gamma_+^\varepsilon \cup\, \Gamma_-^\varepsilon,\;\; \Gamma_0^\varepsilon = \partial\Omega \times (-\varepsilon,\varepsilon),\;\; \Gamma_+^\varepsilon = \Omega \times \{\varepsilon\},\;\; \Gamma_-^\varepsilon = \Omega \times \{-\varepsilon\}.$$

We suppose that the boundary Γ_0^ε (the lateral part of $\partial\Omega^\varepsilon$) consists in two non-overlapping parts Γ_D^ε and Γ_N^ε. On Γ_D^ε (resp Γ_N^ε) displacements u (resp. tractions) are given.

The following assumptions are adopted :

1. For the sake of simplicity, the body is clamped on Γ_D^ε which is assumed to be a non-zero Lebesgue measure part of the boundary $\Gamma_0^\varepsilon \subset \partial\Omega^\varepsilon$, that is to say $u^\varepsilon = 0$ on Γ_D^ε.
2. In addition the body can be subjected to a body force $f^V \in L^2(\Omega^\varepsilon)^3$ (such as gravity).
3. As far as loading is concerned, the upper part, Γ_+^ε, is loaded by a surface force $\ell^\varepsilon \in L^2(\Gamma_+^\varepsilon)$, and bottom part Γ_-^ε is for this moment free of surface load.

Remark 1.1. *This choice of boundary conditions is not restrictive : only very slight changes would result from applying clamping conditions on a non zero measure part of* Γ_0^ε *and stress free conditions on the complementary.*

As the plate is assumed to be clamped on a non-zero Lebesgue measure part of the boundary $\partial\Omega^\varepsilon$ denoted Γ_D^ε, thus the space of admissible displacements is :

$$\mathbb{V}^\varepsilon \;=\; \{\; v \in H^1(\Omega^\varepsilon)^3 \;/\; v \;=\; 0 \;\text{ on}\, \Gamma_D^\varepsilon \;\}. \tag{3}$$

1.3.2 Unilateral boundary conditions imposed and week form

Let us now introduce the static equation with Signorini conditions along the plate. We denote by Γ_C^ε a portion of the boundary of the body which is a candidate contact surface with an outward unit normal vector ν. The actual surface on which the body comes into contact with the obstacle is not known in advance, but is assumed contained in the portion Γ_C^ε of Γ_-^ε.

We assume that the plate motion is limited by a rigid obstacle, located below the plate, denoted by

$$g : \Omega^\varepsilon \longrightarrow \bar{\mathbb{R}} := \mathbb{R} \cup \{-\infty, +\infty\}.$$

So, the displacement is constrained to belong to the convex set $\mathbb{K}^\varepsilon \subset \mathbb{V}^\varepsilon$ given by

$$\mathbb{K}^\varepsilon = \{v \in \mathbb{V}^\varepsilon \;/\; v_\nu \leq g \quad \text{on} \quad \Omega^\varepsilon\}, \tag{4}$$

We define on the boundary of Ω^ε an outward unit normal vector ν and τ a tangential unit vector such (ν, τ) soit un repère direct.

The tridimensionnal contact problem in linear elasticity consists in finding the displacement field u and the reaction λ verifying the strong equations (5) and the contact conditions described hereafter :

$$\begin{cases} -\text{div}\, \sigma(u) = f^V & \text{in} \quad \Omega^\varepsilon \\ \sigma(u)\nu = \ell & \text{on} \quad \Gamma^\varepsilon_+ \end{cases} \tag{5}$$

where σ stands for the stress tensor field and div denotes the divergence operator of tensor valued functions. The notation γ represents the linearized strain tensor field, *i.e.* : $\gamma_{ij}(u) = \frac{1}{2}(\partial_i u_j + \partial_j u_i)$. The Hooke's law for an isotropic material is expressed thanks to the fourth order symmetric elasticity Hooke tensor $\mathbf{S}$ on Ω^ε having the usual uniform ellipticity and boundedness property :$\sigma = \mathbf{S}\gamma$.
We decompose the displacement field $u : \Omega^\varepsilon \longrightarrow \mathbb{R}^3$ and density of surface forces $\sigma(u)\nu : \Gamma^\varepsilon_- \longrightarrow \mathbb{R}$ thanks to (ν, τ) :

$$u = u_\nu \nu + u_\tau \quad \text{and} \quad \sigma(u)\nu = \sigma_\nu(u)\nu + \sigma_\tau(u).$$

The unilateral contact conditions (or classical Kuhn-Tucker conditions) are given by :

$$u_\nu \leq g, \quad \sigma_\nu(u) \leq 0 \quad \text{and} \quad (u_\nu - g)\sigma_\nu(u) = 0 \text{ on } \Gamma^\varepsilon_-. \tag{6}$$

In the frictionless contact case this condition is simply *i.e.*

$$\sigma_\tau(u) = 0, \text{ on } \Gamma^\varepsilon_-. \tag{7}$$

The Coulomb friction condition on Γ^ε_- reads :

$$\begin{cases} \|\sigma_\tau(u)\| \leq \mathscr{F}\sigma_\nu(u) & \text{if } u_\tau = 0 \\ \sigma_\tau(u) = \mathscr{F}\sigma_\nu(u)\dfrac{u_\tau}{\|u_\tau\|} & \text{otherwise .} \end{cases} \tag{8}$$

where $\mathscr{F} \geq 0$ is the Coulomb friction coefficient, and $\|.\|$ stands for the euclidean norm in $\mathbb{R}^2$.

From the Green formula and equations (5), we get the weak variational formulation for every $v \in \mathbb{V}^\varepsilon$:

$$\int_{\Omega^\varepsilon} \sigma(u) : \gamma(v)\, \mathrm{d}\Omega - \int_{\Gamma^\varepsilon_-} \sigma(u) \cdot v\, \mathrm{d}\Gamma = \int_{\Omega^\varepsilon} f^V \cdot v\, \mathrm{d}\Omega + \int_{\Gamma^\varepsilon_+} \ell \cdot v\, \mathrm{d}\Gamma. \tag{9}$$

Remark 1.2. *The Korn's inequality which exists for this problem is simply adapted from the classical one, which makes it possible to prove the existence and uniqueness of the solution to problem (9) submitted to (6) with any fixed* $\varepsilon > 0$, *thanks to the Lax-Milgram lemma. For the general problem (9) submitted to (8) the existence of a solution is available only for small Coulomb friction coefficient* $\mathscr{F} \geq 0$.

2 3D-Contact problems : Variational formulation using Nitsche's method

A reformulation of the previous contact conditions comes from the augmented Lagrangian formulation of contact problems. Let r be a given real positive number. As in [1, 2, 3, 4, 5, 6], the contact conditions (6) are rewritten as :

$$\sigma_\nu(u) = [\sigma_\nu(u) + r(g - u_\nu)]_- \;, \tag{10}$$

where we denote $[x]_- = \begin{cases} 0 & \text{if } x \geq 0 \\ x & \text{otherwise} \end{cases}$.

We could also reformulate the Coulomb friction condition using the simple ball projection $P_{B(0,s)}$ defined by :

$$\mathbf{P}_{B(0,s)}(q) = \begin{cases} \Pi_\nu(q) & \text{if } \|\Pi_\nu(q)\| \leq s, \\ s\dfrac{\Pi_\nu(q)}{\|\Pi_\nu(q)\|} & \text{otherwise,} \end{cases}$$

where the tangential projection tensor is $\Pi_\nu = \mathbf{I} - \nu \otimes \nu$, and $\mathbf{I}$ is the identity tensor.

In fact, for a given positive function r, the friction condition is equivalent to the non-smooth equation :

$$\sigma_\tau(u) = \mathbf{P}_{B(0,-\mathscr{F}[\sigma_\tau(u)+r(g-u_\nu)]_-)}(\sigma_\tau(u) - ru_\tau). \tag{11}$$

Let now $\mathscr{O} \in \mathbb{R}$ be a fixed parameter that we use to recover different variants of the Nitsche method, as in the 3D linear elastic setting (see, e.g., [4]). We inject finally the expressions (10) and (11) into (9) and obtain, formally, our Nitsche's based formulation for frictional contact. In particulier, we get for $\mathscr{O} = 1, 0, -1$ a symmetric, non-symmetric, and anti-symmetric method :

$$\int_{\Gamma_-^\varepsilon} \sigma(u) \cdot v \, d\Gamma = \int_{\Gamma_-^\varepsilon} \sigma_\nu(u) v_\nu(v) \, d\Gamma + \int_{\Gamma_-^\varepsilon} \sigma_\tau(u) \cdot v_\tau \, d\Gamma \tag{12}$$

with the normal contact term (obtained from (10) and the classical decomposition introduced in [4, 6]) :

$$\begin{aligned} \int_{\Gamma_-^\varepsilon} \lambda_\nu v_\nu(v) \, d\Gamma &= \frac{1}{r}\int_{\Gamma_-^\varepsilon} \lambda_\nu \left(r v_\nu - \mathscr{O}\sigma_\nu(v)\right) \, d\Gamma + \frac{\mathscr{O}}{r}\int_{\Gamma_-^\varepsilon} \lambda_\nu \sigma_\nu(v) \, d\Gamma \\ &= \frac{1}{r}\int_\Omega [\lambda_\nu + r(g-u_\nu)]_- \left(r v_\nu - \mathscr{O}\sigma_\nu(v)\right)\big|_{x_3=-\varepsilon} \, d\Gamma + \frac{\mathscr{O}}{r}\int_\Omega \lambda_\nu \sigma_\nu(v)\big|_{x_3=-\varepsilon} \, d\Gamma \end{aligned} \tag{13}$$

and with frictional contact term :

$$\int_{\Gamma_-^\varepsilon} \lambda_\tau \cdot v_\tau \, d\Gamma = \int_{\Gamma_-^\varepsilon} \mathbf{P}_{B(0,-\mathscr{F}[\lambda_\nu+r(g-u_\nu)]_-)}(\lambda - ru_\tau) \cdot v_\tau \, d\Gamma \tag{14}$$

notting that :

$$v_\tau = v - v_\nu \nu = v - (\underline{v}_\alpha \nu_\alpha - x_3 \partial_\alpha v_3 \nu_\alpha + v_3 \nu_3)\nu,$$
$$u_\tau = u - u_\nu \nu = u - (\underline{u}_\alpha \nu_\alpha - x_3 \partial_\alpha u_3 \nu_\alpha + u_3 \nu_3)\nu,$$

Now we insert the expressions (13) and (14) of (12) into variational problem (9), to write the so called Nitsche-based method :

$$\begin{cases} \text{Find a sufficiently regular } u \in \mathbb{V}^\varepsilon \text{ such that for all sufficiently regular } v \in \mathbb{V}^\varepsilon : \\ \displaystyle\int_{\Omega^\varepsilon} \sigma(u) : \gamma(v) \, d\Omega + \frac{1}{r}\int_\Omega [\sigma_\nu(u) + r(g-u_\nu)]_- \left(r v_\nu - \mathscr{O}\sigma_\nu(v)\right)\big|_{x_3=-\varepsilon} \, d\Gamma + \frac{\mathscr{O}}{r}\int_\Omega \sigma_\nu(u)\sigma_\nu(v)\big|_{x_3=-\varepsilon} \, d\Gamma \\ \displaystyle + \int_{\Gamma_-^\varepsilon} \mathbf{P}_{B(0,-\mathscr{F}[\sigma_\nu(u)+r(g-u_\nu)]_-)}(\sigma(u) - ru_\tau) \cdot v_\tau \, d\Gamma = \int_{\Omega^\varepsilon} f^V \cdot v \, d\Omega + \int_{\Gamma_+^\varepsilon} \ell \cdot v \, d\Gamma. \end{cases} \tag{15}$$

3 Discrete Nitsche's variational formulation in Ω^ε

In what follows, Ciarlet's notations [7] are used. Let $\mathscr{T}_h$ be a family of triangulations of the domain Ω^ε such that $\Omega^\varepsilon = \bigcup_{K\in\mathscr{T}_h} K$. Let h_K be the diameter of $K \in \mathscr{T}_h$ and $h = \max_{K\in\mathscr{T}_h} h_K$. The family of triangulations is assumed to be regular, i.e. it exists $C > 0$ such that $\dfrac{h_K}{\rho_K} \leq C$ where ρ_K denotes the radius of the ball inscribed in K. We suppose that the mesh is quasi uniform in the sense that it exists $\zeta > 0$ a constant such that $\forall K \in \mathscr{T}_h,\ h_K \geq \zeta\, h$. We introduce $\mathbb{V}^{\varepsilon,h} \subset \mathbb{V}^\varepsilon$ a family of finite element spaces indexed by h coming from some order $k \geq 1$ finite element method defined on $\mathscr{T}_h$. Furthermore we suppose that this family is conformal to the subdivision of the boundary into Γ_D^ε, Γ_N^ε and Γ_C^ε (i.e., a face of an element $K \in \mathscr{T}_h$ is not allowed to have simultaneous non-empty intersection with more than one part of the subdivision). We choose a standard $\mathscr{P}^k$-Lagrange finite element method of degree k with $k = 1$ or $k = 2$, i.e. :

$$\mathbb{V}^{\varepsilon,h} := \{v \in (\mathscr{C}^0(\bar{\Omega}^\varepsilon))^3 : v_{|K} \in (\mathscr{P}^k(K))^3,\ \forall K \in \mathscr{T}_h,\ v = 0 \text{ on } \Gamma_D^\varepsilon\}.$$

Then, a finite element approximation of our Nitsche-based method (15) reads as :

$$\begin{cases} \text{Find } u^h \in \mathbb{V}^{\varepsilon,h} \text{ such that :} \\ \displaystyle\int_{\Omega^\varepsilon} \sigma(u^h) : \gamma(v^h)\, d\Omega + \frac{1}{r}\int_\Omega \Big[\sigma_\nu(u) + r(g - u^h_\nu)\Big]_- \Big(rv^h_\nu - \mathscr{O}\sigma_\nu(v^h)\Big)\Big|_{x_3=-\varepsilon} d\Gamma + \frac{\mathscr{O}}{r}\int_\Omega \sigma_\nu(u)\sigma_\nu(v^h)\big|_{x_3=-\varepsilon} d\Gamma \\ \displaystyle + \int_{\Gamma_-^\varepsilon} \mathbf{P}_{B(0,-\mathscr{F}[\sigma_\nu(u)+r(g-u^h_\nu)]_-)}(\sigma(u) - ru^h_\tau)\cdot v^h_\tau\, d\Gamma = \int_{\Omega^\varepsilon} f^V \cdot v^h\, d\Omega + \int_{\Gamma_+^\varepsilon} \ell \cdot v^h\, d\Gamma, \qquad \forall v^h \in \mathbb{V}^{\varepsilon,h}. \end{cases} \tag{16}$$

3.1 Variationnal formulation for 2D-bilaplacian plate contact problem and discrete Nitsche's method

In this section, we consider a 2D-bilaplacian Kirchhoff plates constrained by a rigid obstacle. We follow R. STENBERG et al. [11] to implement a Nitsche-type method with only the vertical displacement ($\tilde{u} := u_3 : \Omega \longrightarrow \mathbb{R}$) variable as an unknown. Therefore, we only address this formulation to compare numerical results we obtain below with 3D-models of plate. So, let us first recall the Kirchhoff-Love frictionless contact problem for a thin plate. The variational formulation for a Kirchhoff-Love thin elastic clamped/free plate above an rigid obstacle consists in :

$$\begin{cases} \text{Find } \tilde{u} \in \tilde{\mathbb{K}} \text{ such that for any } \tilde{w} \in \tilde{\mathbb{K}} \\ \displaystyle\int_\Omega \frac{D}{2\varepsilon}\Big[(1-N)\,\partial^2_{\alpha\beta}\tilde{u} + \nu\,\Delta\tilde{u}\,\delta_{\alpha\beta}\Big]\,\partial^2_{\alpha\beta}\tilde{w}\,dx - \int_\Omega \tilde{\sigma}_\nu \tilde{w}dx = \int_\Omega \tilde{f}\,\tilde{w}\,dx, \end{cases} \tag{17}$$

with $\tilde{f} = \displaystyle\int_{-\varepsilon}^{\varepsilon} f_3^V\, dx_3$ is the resulting transverse loading, $\varepsilon > 0$ is assume to be constant all along the plate. And the bending modulus is $D = \dfrac{2\,E\,\varepsilon^3}{3\,(1-N^2)}$, for a plate made of a homogeneous and isotropic material. The plate is assumed to be clamped on a non-zero Lebesgue measure part of the boundary $\partial\Omega$. Then the convex set of admissible displacements is $\tilde{\mathbb{K}} \subset \tilde{\mathbb{V}}$ given by $\tilde{\mathbb{K}} = \{\tilde{w} \in \tilde{\mathbb{V}} \,/\, \tilde{w} \leq g \text{ on } \Omega\}$, and $\tilde{\mathbb{V}} = \{\, \tilde{w} \in H^2(\Omega) \,/\, \tilde{w}(x) = 0 = \partial_n \tilde{w}(x) ,\ \forall x \in \Gamma_c \,\}$, where $\partial_n \tilde{w}$ is the normal derivative along Γ_c .

Let $\tilde{\mathscr{T}}_h$ be a conforming shape regular triangulation of Ω which we assume to be polygonal, and the finite element subspace associated is $\tilde{\mathbb{V}}^{\varepsilon,h} \subset \tilde{\mathbb{V}}$. The approximation properties of the primal variable and the Nitsche term are balanced when the polynomial order of the latter is four degrees smaller than that of the displacement variable, for example, when the Argyris element is coupled with a piecewise linear and discontinuous approximation of the Lagrange multiplier. A reformulation of the normal contact conditions with a Nitsche-type method consists in implicit the discret Lagrange multiplier from the previous formulation of contact problem (17). As the contact reaction is a measure possibly singular, that is why we consider, as in [11], the following L^2-approximation of the reaction force :

$$\tilde{\sigma}^{\pi^h}_\nu(u)\Big|_K = \frac{1}{\kappa + \tilde{r}h_K^4}\Big[\pi^h(g)_{|K} - \pi^h(\tilde{u}^h)_{|K} + \tilde{r}h_K^4(\pi^h(\Delta^2\tilde{u}^h) - \pi^h(\tilde{f}))\Big]_- \quad \forall K \in \tilde{\mathscr{T}}_h \tag{18}$$

where π^h is the L^2-projection, $\kappa \geq 0$ is the classical penalization parameter, and $\tilde{r} \geq 0$ is the stabilisation or Nitsche parameter. This hybrid formulation (18) of the Lagrange multiplier corresponds to a pure penalisation approch when $\tilde{s} = 0$, and a pure Nitsche's method if $\kappa = 0$. We introduce the function $H : \Omega \longrightarrow \mathbb{R}$ such that $H\big|_K = h_K$, for all $K \in \tilde{\mathscr{T}}_h$. Thus, a finite element approximation of (17) based on hybrid Nitsche method (18) is :

$$\left\{\begin{array}{l}\text{Find } \tilde{u}^h \in \tilde{\mathbb{V}}^h \text{ such that :}\\ \tilde{a}^h(\tilde{u}^h,\tilde{v}^h;\tilde{u}^h) = \tilde{L}^h(\tilde{v}^h;\tilde{u}^h) \qquad \forall \tilde{v}^h \in \tilde{\mathbb{V}}^h.\end{array}\right. \tag{19}$$

where

$$\tilde{a}^h(\tilde{u}^h,\tilde{v}^h;\tilde{u}^h) = \tilde{a}(\tilde{u}^h,\tilde{v}^h) + \int_{\Omega_C(\tilde{w}^h)} \frac{1}{\kappa+\tilde{r}H^4}\,\tilde{u}^h\,\tilde{v}^h\,dx - \int_{\Omega_C(\tilde{w}^h)} \frac{\tilde{r}H^4}{\kappa+\tilde{r}H^4}\,D\Delta^2(\tilde{u}^h)\,\tilde{v}^h\,dx$$

$$-\int_{\Omega_C(\tilde{w}^h)} \frac{\tilde{r}H^4}{\kappa+\tilde{r}H^4}\,D\Delta^2(\tilde{v}^h)\,\tilde{u}^h\,dx - \int_{\Omega_C(\tilde{w}^h)} \frac{\kappa\tilde{r}H^4}{\kappa+\tilde{r}H^4}\,D^2\Delta^2(\tilde{u}^h)\,\Delta^2\tilde{v}^h\,dx$$

$$-\int_{\Omega\setminus\Omega_C(\tilde{w}^h)} \kappa H^4\,D^2\Delta^2(\tilde{u}^h)\,\Delta^2\tilde{v}^h\,dx$$

$$\tilde{L}^h(\tilde{v}^h;\tilde{u}^h) = \int_\Omega \tilde{f}\,\tilde{v}^h\,dx + \int_{\Omega_C(\tilde{w}^h)} \frac{1}{\kappa+\tilde{r}H^4}\,g\,\tilde{v}^h\,dx$$

$$-\int_{\Omega_C(\tilde{w}^h)} \frac{\tilde{r}H^4}{\kappa+\tilde{r}H^4}\,D\Delta^2(\tilde{v}^h)\,g\,dx - \int_{\Omega_C(\tilde{w}^h)} \frac{\tilde{r}H^4}{\kappa+\tilde{r}H^4}\,f\,\tilde{v}^h\,dx - \int_{\Omega_C(\tilde{w}^h)} \frac{\tilde{r}\kappa H^4}{\kappa+\tilde{r}H^4}\,D\Delta^2(\tilde{v}^h)\,f\,dx$$

$$-\int_{\Omega\setminus\Omega_C(\tilde{w}^h)} \kappa H^4\,D\tilde{f}\,\Delta^2\tilde{v}^h\,dx$$

The contact set $\Omega_C(\tilde{w}^h)$ above is given by $\Omega_C(\tilde{w}^h) = \{(x,y) \in \Omega : \tilde{\sigma}^h_\nu(\tilde{w}^h) > 0\}$, with $\tilde{\sigma}^h_\nu(\tilde{w}^h)$ denoting the discrete reaction force given by $\tilde{\sigma}^h_\nu(\tilde{w}^h) = \frac{1}{\kappa+\tilde{r}H^4}\left[g - \tilde{w}^h + \tilde{r}H^4(\Delta^2\tilde{w}^h - \tilde{f})\right]_-$.

The practical solution algorithm for Problem (16) is a fixed point process where at each step the contact set Ω_C is approximated using the displacement field from the previous iteration so that system (16) becomes linear. The process is terminated as soon as the norm of the displacement field is below a predetermined tolerance TOL > 0. The stopping criterion is formulated with respect to the strain energy norm $\|w\|_{\tilde{a}} = \sqrt{\tilde{a}(w,w)}$.

4 Numerical results

We end this note with numerical results in order to test differents models. The corresponding test-case concerns a plate $\Omega = [0,10]\times[0,10]$ clamped along its boundary $\partial\Omega$, of thickness $2\varepsilon = 0.01$mm. with Young modulus $E = 2\times 10^5$ GPa, and Poisson ratio $N = 0.3$. The plate motion is above a rigid obstacle, $x_3 = -1$, and the plate is submited to a vertical constant and homogenous forces $f^V = (0,0,-0.02/2\varepsilon)$ GPa in 3D and $\tilde{f} = -0.02$ GPa, in 2D. We compare our results as on Figure 3 with (16) and those obtained with the formulation (19) from [11] on Figure 2 or those obtained with the classical Nitsche 3D method on Figure 1.

Other numerical results will be presented during the congress, in particular an analysis of convergences of the various approaches proposed above.

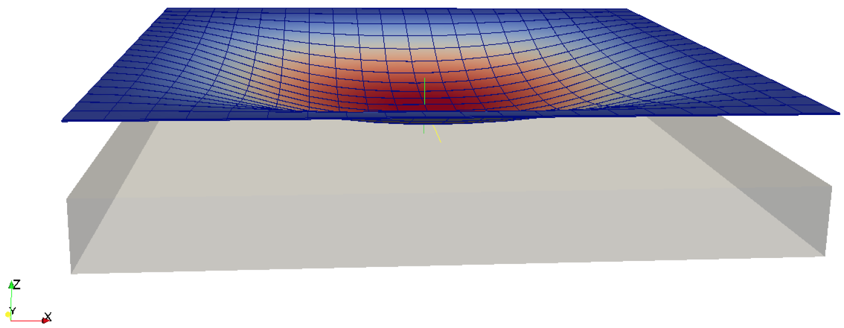

FIGURE 1 – Solution 3D with classical Nitsche method

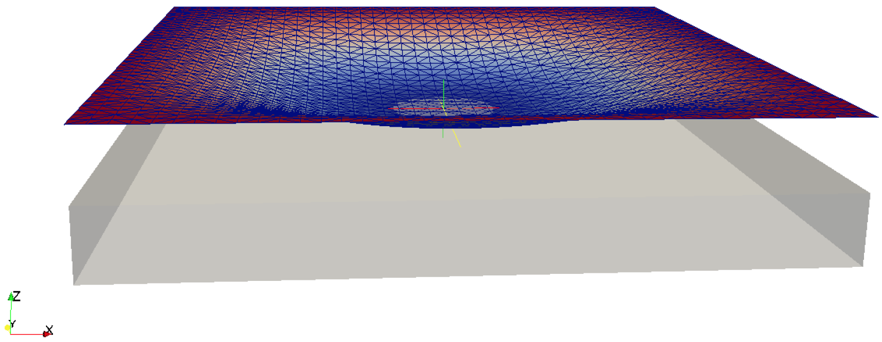

FIGURE 2 – Solution 2D Kirchoff-Love with STENBERG method

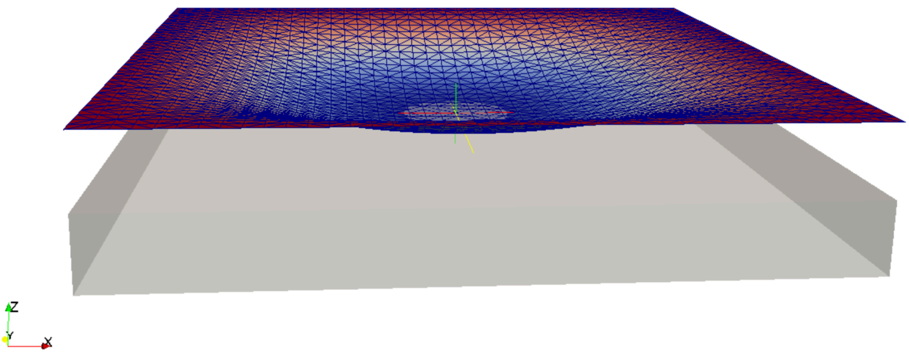

FIGURE 3 – Solution 2D Kirchoff-Love with Nitsche method

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