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An eXtended-Newton Scheme for the Solution of Stiff Normal Contact Problems

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Résumé — The leading idea to the algorithm presented in this work could be used to solve hard nonlinear problems ; these that can hardly be solved by classical schemes. This idea is herein explained for the solution of a class of stiff normal contact problems between two solids. For these, it consists in a scheme that constructs, iteratively, approximate contact models, leading to softer, still nonlinear, problems, better suited to classical schemes, whose convergent solution solves the stiff contact problem. Numerical results show the remarkable performance of this algorithm, labeled extended-Newton scheme.

Mots clés — Hard nonlinear problems, Stiff normal contact, eXtended-Newton Algorithm.

1 Introduction

A necessary convergence condition of the phenomenal Newton's algorithm, when used to find out a solution of a nonlinear and regular problem is to initialize the algorithm "sufficiently" near from a solution of the problem. Moreover, the rate of convergence of this method is linked to the problem conditioning. These are probably the main reasons for which this method is not well-suited for the solution of hard nonlinear problems, such as stiff (thus ill-conditioned) contact problems.

A concept of iteration on models is presented in this work. This could be used to construct schemes able to solve certain hard nonlinear problems, namely these that can hardly be solved by classical algorithms. The main ideas behind this concept are herein specified and applied for the solution of a class of stiff normal contact problems between two solids.

Normal contact problems vary according to the fineness sought on the quantities of interest at the interface of these solids. It is common to classify them in two categories, namely, those (macroscopic) based on a non-interpenetration geometric consideration and those introducing constitutive relationships to account for micro phenomena at the interface (asperities deformations, ...). For the last ones, whether they are purely repulsive as compliant, exponential or barrier models (see e.g [6, 8, 9] and the references mentioned in these monographs) or repulsive and attractive as Lennard-Jones model [7] (with a relatively small Tabor parameter [10]), these models give rise to stiff mechanical nonlinear contact problems. As they are nonlinear, their resolution requires iterative algorithms. However, it is well known that the important stiffness of these contact models poses great convergence difficulties to classical prediction-correction algorithms, including the *phenomenal Newton's algorithm*. The purpose of this contribution is to reveal a new strategy that introduces a concept of Algorithm of models, taking a cue and extending the Newton idea, to overcome the stiffness difficulty. For this and for the sake of illustration, we address here the case of a normal stiff compliant contact problem, denoted by $\mathcal{P}_c$

Let us first mention that to ease the solution of $\mathcal{P}_c$, an approximate model to a full compliant contact model, labeled (slightly abusive) multi-compliant model, has been introduced in [1] to define an approximate problem for $\mathcal{P}_c$ by linearizing the full compliant contact model, defined on the potential contact surface Γ_c, beyond a given threshold field, we denote here $\mathcal{T}$. The main interest of this linearization was to have a contact problem easier to solve (because less stiff), with classical algorithms (and with Newton's method, in particular) than $\mathcal{P}_c$, while being an approximation of the latter. Let us also observe that, by following the same purpose, an alternative multilevel compliant/Signorini contact model has also been suggested, by following similar lines [2] (see also [3]). However, though interesting in practice, there is a lack of control of the quality of the approximate solutions of these surrogate models.

The aim of this communication consists in obtaining the full solution of the considered full compliant contact problem, with a pre-defined precision. For this, the mainstream idea of this contribution consists in pushing forward the contact model approximations ideas, recalled above, by introducing the following iterative scheme :

— We consider the threshold field $\mathcal{T}$ as an initial threshold field, denoted $\mathcal{T}^0$ and we denote $\mathcal{M}_c^0$ the approximate compliant contact model, based on $\mathcal{T}'$.
— We solve the approximate contact problem $\mathcal{P}_c^0$ (obtained by approximating the full compliant contact model used in $\mathcal{P}_c$ by $\mathcal{M}_c^0$ and obtain the approximate solution $\mathcal{S}^0$.
— We update appropriately $\mathcal{T}_c^0$, by using $\mathcal{S}^0$, to construct the new threshold field $\mathcal{T}_c^1$.
— We calculate the relative model-error. If a convergence criteria is met then we stop iterating. Otherwise, our scheme is continued by following exactly the same steps as described above, till convergence.

The most important point here is that this upstream level of model iterations leads to simplified (less hard), but still nonlinear approximate problems that can hopefully be solved by appropriate classical algorithms ; the Newton's method being the most appropriate candidate whenever it is usable.
Actually, we claim that, conceptually, our mainstream idea of introduction of iterations on models to get the solution of a given model, extendes the fundamental Newton's Algorithm idea. Indeed, let us recall that in Calculus of Variation Theory, when one wants to find out the/a minimizer of a sufficiently regular non quadratic energy $\mathcal{J}$ on a vector space (a situation encountered in many nonlinear mechanical and physical problems), the Newton's method is often used to find out the/a solution that satisfies the necessary Euler non linear Identity, associated to the optimization problem. When applied in this framework, the Newton's method is often considered as an iterative second-order solver : at each iteration n, being given a current approximate solution u_n (implicitly assumed to be sufficiently close from the/a minimizer solution), to decrease the energy, a second order-based descent direction is used. More precisely, having initialized the process and knowing $\underline{u}_n$) which does not meet a given convergence criterion, the next iterate $\underline{u}_{n+1}$ is obtained by :

$$\begin{cases} \underline{u}_{n+1} = \underline{u}_n - \rho \underline{d}_n \\ \underline{d}_n = (\underline{\underline{\mathcal{H}}}(\underline{u}_n))^{-1} \underline{\nabla} \mathcal{J}(\underline{u}_n) \end{cases} \tag{1}$$

where $\rho > 0$ stands for a given path (often equal to 1 and where $(\underline{\underline{\mathcal{H}}}(\underline{u}_n))$ and $\underline{\nabla}\mathcal{J}(\underline{u}_n)$ stand for the Hessian Matrix and the Gradient vector of the energy functional $\mathcal{J}$ taken at $\underline{u}_n$, respectively.

However, it is worthy remembering that the algorithm (1) relies more basically on an iterative linear models approximations of the nonlinear Euler system, by using, at each iteration, a first order Taylor expansion for the Euler equations at $\underline{u}_{n+1} = \underline{u}_n + (\underline{u}_{n+1} - \underline{u}_n))$ (assuming that $\underline{u}_{n+1} - \underline{u}_n$ is sufficiently small). Having these elements in mind, by approximating the solution of a hard nonlinear problem by a scheme constructing and solving iteratively simplified soft problems, we use a similar concept to the newton's idea, with a significant difference : the simplified models are still nonlinear (but softer than the targeted problem). This explains the title of this contribution.

The global methodology is dmathcalTiled in the remainder of this work for the solution on stiff compliant contact model by means of solutions of a sequence of multilevel compiant/Signorini contact approximation models. Global solution strategies are dmathcalTiled and assessed numerically.

2 Formulations of a two-body contact problem based on a normal compliant model

Consider the frictionless contact between two elastic solids $\mathcal{B}_i$ ($i = 1, 2$), each of which occupying the closure of a bounded domains Ω_i in $\mathbb{R}^d$ ($d = 2$ or 3). A body force $\boldsymbol{f}_i$ is exerted in each domain. Suppose that the boundary of Ω_i is partitioned into a clamped part Γ_{iu}, a part Γ_{it} where a surface traction

$\boldsymbol{t}_i$ is prescribed and a part Γ_{ic} where contact may occur. The unit outward normal vector at each boundary is denoted by $\boldsymbol{n}_i$. Under small perturbation hypotheses, the local boundary-value problem governing the behavior of the considered mechanical system reads :

$$\text{for } i = 1,2, \begin{cases} div(\boldsymbol{\sigma}_i) + \boldsymbol{f}_i = \boldsymbol{0} & \text{in } \Omega_i \\ \boldsymbol{\sigma}_i = \boldsymbol{R}_i \boldsymbol{\varepsilon}_i & \text{in } \Omega_i \\ \boldsymbol{\varepsilon}_i = \frac{1}{2}(\nabla \boldsymbol{u}_i + (\nabla \boldsymbol{u}_i)^T) & \text{in } \Omega_i \\ \boldsymbol{u}_i = \boldsymbol{0} & \text{on } \Gamma_{iu} \\ \boldsymbol{\sigma}_i \boldsymbol{n}_i = \boldsymbol{t}_i & \text{on } \Gamma_{it} \\ \boldsymbol{\sigma}_i \boldsymbol{n}_i = p_{in} \boldsymbol{n}_i & \text{on } \Gamma_{ic} \end{cases} \tag{2}$$

where $\boldsymbol{u}_i$, $\boldsymbol{\sigma}_i$, $\boldsymbol{\varepsilon}_i$ and $\boldsymbol{R}_i$ denote the displacement field, the stress tensor, the linearized strain tensor and the elasticity moduli in Ω_i ; p_{in} denotes the value of the normal contact pressure on Γ_{ic}.

For the completeness of the formulation, we adopt the classical *master-slave* distinction of the two contacting surfaces and note $\Gamma_c = \Gamma_{1c}$, $p_n = p_{1n} = -p_{2n}$ and $\boldsymbol{n} = \boldsymbol{n}_1 = -\boldsymbol{n}_2$. Then we define the full compliant contact model by :

$$p_n = p_c = -\kappa((d_n - d_0)^+)^m \tag{3}$$

where $\kappa > 0$ and $m > 1$ are parameters of the compliant contact model that are identified experimentally (see e.g. [6]). Let us mention that according to the usual range of this parameter is $2 \leq m \leq 3.3$ (see e.g. Kragelskii et al., 1965, and the references mentioned therein)

A weak primal formulation of the local compliant contact problem $(2)-(3)$ can readily be obtained. It defines the problem $(\mathcal{P}_c)$ (mentioned in the introduction). The latter can be shown to be equivalent to the problem consisting in minimizing a total energy $\mathcal{J}_c$ on the space of admissible displacements fields $\mathbb{V}_1 \times \mathbb{V}_2$, composed of vector-valued fields, defined in Ω_i, $i = 1,2$, having a $H^1(\Omega_i)$ Sobolev space regularity and satisfying the essential boundary conditions on Γ_{ui}. More importantly, it can be proved that, under classical hypotheses on the data (these used for a linear elasticity problem) and thanks to the convexity of the energy functional $\mathcal{J}_c$, one can check that, for all compliant parameter $m > 1$ (with $d = 2$, case of bidimentional problems) and for $1 < m \leq 3$ (with $d = 3$, case of three-dimensional problems), the energy $\mathcal{J}_c$ is weakly lower semi-continuous on the Banach (here Hilbert) space $\mathbb{V}_1 \times \mathbb{V}_2$ and coercive. Thus, this functional attains its minimum value on $\mathbb{V}_1 \times \mathbb{V}_2$. This means that there exists at least one solution for the problem $(\mathcal{P}_c)$. Moreover, since one can easly check that the energy $\mathcal{J}_c$ is strictly convex, this solution is unique.

The question now is how one can have this solution, or at least a family of approximating solutions by using computational tools ? (the main difficulty being the hard nonlinear character of the compliant contact problem) ? This issue is handled in the next section.

3 Formulations of a two-body contact problem based on a normal compliant/Signorini model

The more the power m in the nonlinear compliant contact model is large, the more the related contact problem $(\mathcal{P}_c)$ is stiff, leading to ill-condioned problems, thus hard to be handled by classical algorithms. To handle this issue, a first step was achieved in[2, 3] where a multi-level compliant/Signorini model was introduced to approximate the full compliant one, defined by (3) . The former, denoted by $\mathcal{M}_J^\prime$ is defined by :

$$\begin{cases} p_n = \mathcal{S}_G \lambda + \mathcal{S}_G p_c(-d_0) + (1 - \mathcal{S}_G) p_c, \text{ on } \Gamma_c \\ \lambda = \mathcal{S}_G(\lambda - \rho d_n), \ p_c = -\kappa((d_n - d_0)^+)^m, \text{ on } \Gamma_c \\ \mathcal{S}_G = \mathbb{1}_{R^-}(\lambda - \rho d_n), \text{ on } \Gamma_c \end{cases} \tag{4}$$

where p_c and λ denote the contact pressure of the compliant model and the Signorini model, respectively, $\mathcal{S}_G$ denotes the *level-set* or status field characterizing the contact/non-contact state for the Signorini model, $\rho_n > 0$ is a homogenization parameter, $\mathbb{1}_{\mathcal{K}}$ is the characteristic function of the set $\mathcal{K}$ and d_n is the signed-distance between the contacting (averaged rough) surfaces. The term $\mathcal{S}_G p_c(-d_0)$ in equation (4) is introduced ton ensure the continuity of the contact pressure in transition points from compliant to Signorini contact interface response.

Let us here observe that the Signorini model used in (4) corresponds to the classical macroscopic one ($d_n \leq 0$, $\lambda \leq 0$ and $d_n\lambda = 0$, on Γ_c).
As for the full compliant model, one can derive the weak formulation of the local multi-level compliant/Signorini contact problem by using (2) and (4), denoted ($\mathcal{P}_c^0$). It can also be shown that this problem has a unique solution u_c^0. For general (acceptable) external loadings acting on the two solids, u_c^0 is only an "approximation" of the solution of ($\mathcal{P}_c$).

4 An eXtended Newton's algorithm

Our aim is now to compute the solution of the full compliant contact problem, with a pre-defined precision. For this, based on the approximation defined in the previous section, we define an algorithm of models that relies on an idea that, conceptually, could be seen as an extension of the Newton's algorithm fundamental idea.

4.1 Description of the algorithm of models

Considering the model defined by (4) as an initial Compliant/Signorini contact model, characterized by an initial threshold field $\mathcal{T}^0 = 0$, on Γ_c, we construct a sequence ($N = 0, 1, 2...$) of multilevel compliant/Signorini contact models, characterized by a threshold field $\mathcal{T}^N$. The N-th contact model reads :

$$\begin{cases} p_n^N = \mathcal{S}_G^N \lambda^N + \mathcal{S}_G^N p_c(-d_0 + \mathcal{T}^N) + (1 - \mathcal{S}_G) p_c^N, \text{ on } \Gamma_c \\ \lambda^N = \mathcal{S}_G^N(\lambda^N - \rho(d_n^N - \mathcal{T}^N)),\ p_c^N = -\kappa((d_n^N - d_0)^+)^m \\ \mathcal{S}_G^N = \mathbb{1}_{R^-}(\lambda^N - \rho(d_n^N - \mathcal{T}^N)) \end{cases} \tag{5}$$

The hybrid weak formulation of the local contact problem, defined by (2) and (5), denoted by ($\mathcal{P}_c^N$) reads : (see [5])

Find $(\boldsymbol{u}_1^N, \boldsymbol{u}_2^N, \lambda^N, \mathcal{S}_G^N) \in \mathbb{V}_1 \times \mathbb{V}_2 \times \mathbb{H} \times L^\infty(\Gamma_c)$; $\forall (\boldsymbol{v}_1, \boldsymbol{v}_2, \mu) \in \mathbb{V}_1 \times \mathbb{V}_2 \times \mathbb{H}$,

$$\begin{cases} \sum_{i=1,2} G_i^{int}(\boldsymbol{u}_i^N, \boldsymbol{v}_i) - \sum_{i=1,2} G_i^{ext}(\boldsymbol{v}_i) - G^c(\boldsymbol{u}_1^N, \boldsymbol{u}_2^N, \boldsymbol{v}_1, \boldsymbol{v}_2, \lambda^N, \mathcal{S}_G^N) = 0 \\ -\frac{1}{\rho} \int_{\Gamma_c} (\lambda^N - \mathcal{S}_G^N(\lambda - \rho(d_n^N - \mathcal{T}^N)))\mu = 0 \end{cases}$$

where $\mathbb{H}$ denotes the space of the Lagrange multipliers of the Signorini model ; G_i^{int}, G_i^{ext} and G^c denote the virtual works of internal-, external- and contact forces, respectively, defined by :

$$G_i^{int}(\boldsymbol{u}_i^N, \boldsymbol{v}_i) = \int_{\Omega_i} \boldsymbol{\sigma}_i(\boldsymbol{u}_i^N) : \boldsymbol{\varepsilon}_i(\boldsymbol{v}_i)$$

$$G_i^{ext}(\boldsymbol{v}_i) = \int_{\Omega_i} \boldsymbol{f}_i \cdot \boldsymbol{v}_i + \int_{\Gamma_{it}} \boldsymbol{t}_i \cdot \boldsymbol{v}_i$$

$$G^c = \int_{\Gamma_c} (\mathcal{S}_G^N(\lambda^N - \rho(d_n^N - \mathcal{T}^N)) + \mathcal{S}_G^N p_c(\mathcal{T}^N) + \mathcal{S}_L^N \kappa(d_n^N - d_0)^m)(\boldsymbol{v}_1 - \boldsymbol{v}_2) \cdot \boldsymbol{n}$$

Once the problem $\mathcal{P}_N$ is solved, the new threshold field $\mathcal{T}^{N+1}$ is obtained as following :

$$\forall \boldsymbol{x} \in \Gamma_c, \begin{cases} \mathcal{T}^{N+1}(\boldsymbol{x}) = \mathcal{T}^N(\boldsymbol{x}) & \text{if } \mathcal{S}_G^N(\boldsymbol{x}) = 0 \\ \mathcal{T}^{N+1}(\boldsymbol{x}) = \underset{\mathcal{T}}{solve}\{p_c(\mathcal{T}) - p_n^N(\boldsymbol{x}) = 0\} & \text{otherwise} \end{cases} \tag{6}$$

The iteration procedure is continued till the following convergence criterion is met :

$$\underset{\boldsymbol{x} \in \Gamma_c}{max} \frac{|\mathcal{T}^{N+1}(\boldsymbol{x}) - \mathcal{T}^N(\boldsymbol{x})|}{|\mathcal{T}^N(\boldsymbol{x})|} < \varepsilon \tag{7}$$

where ε is a small tolerance parameter.

4.2 Discussion of the convergence of the extended Newton algorithm

Let us first establish the following proposition.

Proposition 4.1 *$\forall \boldsymbol{x} \in \Gamma_c$, the sequence of shift-parameters $\mathcal{T}^N(\boldsymbol{x})$, initialized with $\mathcal{T}^0(\boldsymbol{x}) = -d_0$ and updated iteratively, by equation 6 is an increasing sequence.*

Proof 4.2 *We have just to show that, for all $\boldsymbol{x}$ and all N, $\mathcal{T}^N(\boldsymbol{x}) \leqslant \mathcal{T}^{N+1}(\boldsymbol{x})$.*

If, at a point $\boldsymbol{x} \in \Gamma_c$, the Signorini model is not active, i.e. $\mathcal{S}_G^N(\boldsymbol{x}) = 0$, then, according to the first equation of (6), $\mathcal{T}^{N+1}(\boldsymbol{x}) = \mathcal{T}^N(\boldsymbol{x})$. Otherwise, the Signorini model is active, i.e. $\mathcal{S}_G^N(\boldsymbol{x}) = 1$, and one has :

$$\begin{aligned} p_n^N(\boldsymbol{x}) &= \mathcal{S}_G^N(\boldsymbol{x})\lambda^N(\boldsymbol{x}) + \mathcal{S}_G^N(\boldsymbol{x})p_c(\mathcal{T}_N(\boldsymbol{x})) + (1-\mathcal{S}_G^N(\boldsymbol{x}))p_c^N \\ &= \lambda^N(\boldsymbol{x}) + p_c(\mathcal{T}_N(\boldsymbol{x})) \\ &\leqslant p_c(\mathcal{T}_N(\boldsymbol{x})) \end{aligned} \tag{8}$$

where the last inequality is due to the fact that $\lambda^N(\boldsymbol{x}) \leqslant 0$. Considering the fact that the compliant model p_c is decreasing, the inequality in equation (8) implies $\mathcal{T}^N(\boldsymbol{x}) \leqslant \mathcal{T}^{N+1}(\boldsymbol{x})$, by definition of $\mathcal{T}^{N+1}(\boldsymbol{x})$ in (6) when $\mathcal{S}_G^N(\boldsymbol{x}) = 1$.

To conclude that the sequence $(\mathcal{T}^N(\boldsymbol{x}))$, for all $\boldsymbol{x}$ in Γ_c is convergent, we need to show that it is bounded. For this, it seems reasonable to assume that, given external loads acting on the two solids, the density of contact loads for the full compliant model is bounded, implying that the constructed sequence of $(\mathcal{T}^N(\boldsymbol{x}))$ is necessarily bounded.

One can now conclude that, by construction of the fields threshold sequence, at convergence of this sequence, we obtain the solution of the full compliant contact problem.

4.3 Discretization and numerical solution strategies

Each problem $\mathcal{P}_N$ is discretized by means of standard finite element method (FEM) and a collocation method intimately linked to the appropriate numerical integration, taking into account the irregular character of the Status field and the possible incompatibilities of meshed faces of the contact intefaces. For these discrete nonlinear system, a combination of the Newton algorithm and a fixed point strategy on contact Status is used (see e.g. [5, 4]).

The global solution strategy of the proposed extended-Newton methodology is summarized in algorithm 1.

Algorithm 1 An eXtende-Newton method to solve compliant contact problems

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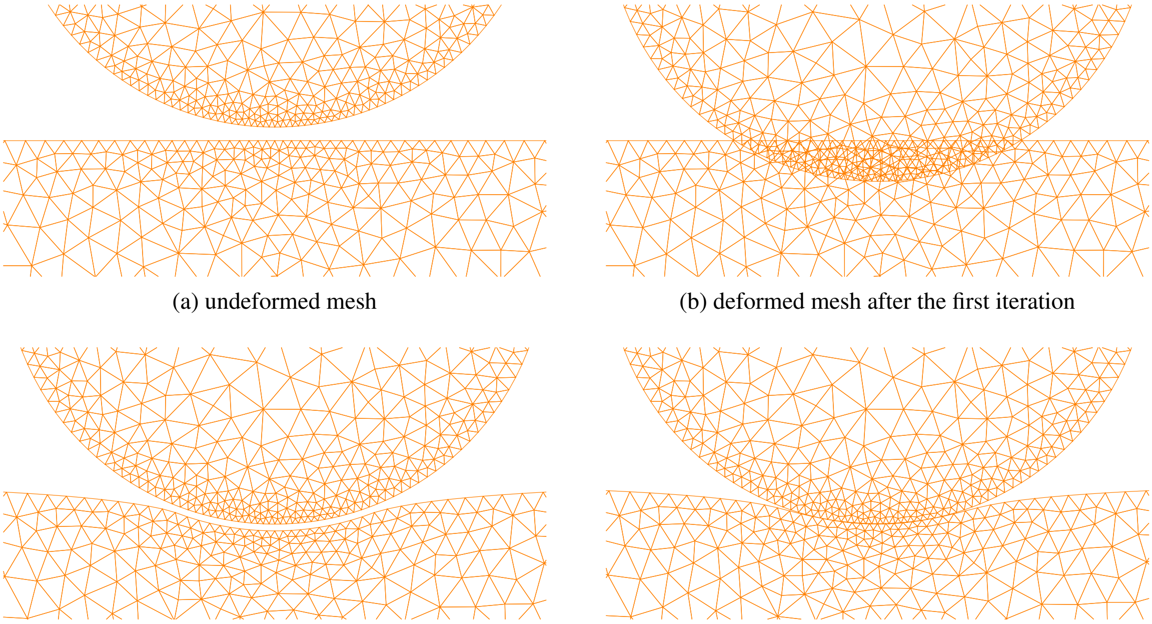
1:  loop over the shift parameter field T^N(x) : N = 0,1,2... with T^0(x) = d_0.
2:      Initialization of problem P_N with the solution of P_{N-1}.
3:      loop over generalized Newton iterations, k = 1,2,3...
4:          solve the linearized problem (equation ??).
5:          update the displacement fields, the Lagrange multiplier field and the status fields.
6:          check convergence of the generalized Newton loop.
7:      end loop
8:      adjust the shift-parameter field with equation 6.
9:      check convergence of the adaptation procedure with equation 7.
10: end loop
```

5 Numerical test

Consider the frictionless contact between a rigid semi-disc with radius $R = 10$ and a rectangular elastic solid of dimension 40×20. The latter is assumed to be isotropic and homogeneous with Young's modulus $E = 100$ and Poison's ratio $\nu = 0.3$. The model parameters for the compliant model and the Signorini model are : $\kappa = 1000$, $d_0 = -0.25$, $m = 3$ and $\rho = 100$, respectively. The initial separation between the two bodies is 0.5. A displacement $u_0 = 2.5$ in the downward direction is prescribed at the top boundary of the semi-disc. The lower boundary of the elastic body is clamped.

By using the proposed eXtended-Newton method, the full-compliant solution is obtained after only 3 adaptations with 12 Newton iterations in total. This is very efficient compared to a classical resolution with many small incremental steps, due to a global ill-conditioning of the discrete problem. The deformed meshes are shown in figure 1 at different solution steps, where both the elastic and rigid bodies are meshed in order to test our method in a more general circumstance. It can be seen that thanks to the incorporation of the shifted-Signorini model, our algorithm allows for large incremental steps (a single increment step in this test), even in cases where large inter-penetrations may occur (see figure 1b for a case where classical compliant model will probably fail due to ill-conditioned tangent matrix).



(a) undeformed mesh

(b) deformed mesh after the first iteration

(c) deformed mesh after the first adaptation

(d) converged mesh

FIGURE 1 – Different solution states of a two-body contact test by using the model-adaptivity method

Figure 2 depicts the contact pressure distribution of the converged solution, where the solution obtained with the full-compliant model and small incremental steps is also presented as a reference solution. The accuracy of the proposed model-adaptivity method can be readily seen.

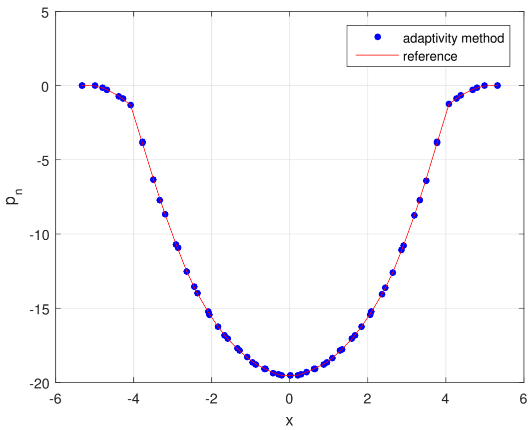


FIGURE 2 – Contact pressure distribution (converged solution)

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